Journal of Statistical Physics, Vol. 47, Nos. 5/6, 1987

Spin Glasses with Long-Range Interaction at High Temperatures

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Received October 23, 1986; revision received April 1, 1987

We study the Ising and N-vector spin glasses with exchange couplings $J \equiv (J_{ij}; i, j \in Z^d)$, which are independent random variables with $EJ_{ij} = 0$ and $EJ_{ij}^n \leqslant \gamma^n n! \, |i-j|^{-n\alpha d}$, for $n \in \mathbb{N}$, some finite constant $\gamma > 0$, and $\alpha > \frac{1}{2}$. For sufficiently small β, we show that for E-a.a. J there is a weakly unique, extremal, infinite-volume Gibbs measure $\mu_{\beta,J}$ for which the expectation of a single (component of) spin vanishes and which has the cluster property in $L_2(E)$ with the same decay as interaction. This work is based on results and methods of Fröhlich and Zegarliński.

KEY WORDS: Spin glasses; long-range interaction; high-temperature region; upper expansion method.

In this note we describe some recent results on spin glasses with arbitrary (admissible) long-range interactions on the lattice $\Gamma \equiv Z^d$. These results have been obtained in Refs. 1 and 2 or are proven with the use of the general methods developed there. First we present a model considered by us and extend a result from Ref. 3 about the existence of the thermodynamic limit for pressure. (A simple proof is given in Appendix A.) Then we discuss the Gibbs structure for the theory of spin glasses with long-range interactions. In particular we consider the problem of constructing the family of local specifications, the relation between strong and weak uniqueness, and the problem of verifying whether an infinite-volume measure satisfies a corresponding DLR equation.

The first main result—existence, weak uniqueness, and extremality of Gibbs measures in the high temperature reqion—is presented in Proposition 1. To give an idea of our upper expansion method we

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sketch—following Refs. 1 and 2—the proofs of weak uniqueness and existence of the infinite-volume measures. Additionally, we give here a general (independent of temperature) method of verifying if the infinite-volume measures satisfy the DLR equation, i.e., if they are Gibbs measures. Using an expansion analogous to one in Ref. 2, we prove also the extremality of our Gibbs measures at high temperatures.

Then we present results concerning the clustering; see Corollary 1 and Proposition 2.

Finally we show (see Appendix C) that our upper expansion method can be applied also to nonrandom systems.

The system we consider is defined by a Hamiltonian function

$$H(\sigma, J) := - \sum_{i,j \in \Gamma, i \neq j} J_{ij} \sigma_i \cdot \sigma_j \tag{1}$$

$\sigma \equiv (\sigma_i \in S_N \cup \{0\},\ i \in \Gamma)$; for $N=1$ we assume $S_N \equiv \{-1, +1\}$. The couplings $J \equiv (J_{ij} \in \mathbb{R}, i, j \in \Gamma)$ are independent random variables subjected to the following conditions, for any $i, j \in \Gamma$:

(Ci) The mean-zero condition

$$EJ_{ij} = 0$$

(Cii) The factorial growth condition

$$|EJ_{ij}^n| \leqslant \gamma^n n!\, |i-j|^{-n\alpha d}, \qquad n \in \mathbb{N}$$

where E is an expectation value, $\alpha > \frac{1}{2}$, and γ is a positive constant.

A finite-volume pressure for H is defined by

$$p_\Lambda(J) := \frac{1}{|\Lambda|} \ln \mu_0 e^{-\beta H_\Lambda} \tag{2}$$

with a product probability measure μ_0 on a configuration space of spins $\Omega \equiv ((S_N \cup \{0\})^\Gamma, \Sigma)$ with the σ-algebra generated by the product topology,

$$H_\Lambda \equiv H(\sigma_i \equiv 0, i \in \Lambda^c) \qquad \text{and} \qquad \beta > 0$$

Assuming (C) and translation invariance of E, then the existence and independence of J, E-a.e., of the infinite-volume pressure

$$p := \lim_{\mathscr{F}_0} p_\Lambda(J) \tag{3}$$

have been proven[3,4] ($\mathscr{F}_0 \equiv \{\Lambda_n \in \Gamma\}_{n\in\mathbb{N}}$ is an increasing Fisher sequence of bounded sets invading Γ). In fact, for this a weaker condition is sufficient with $n \leqslant 4$ in (Cii) (see Appendix A)

Let us now define the finite-volume measures. For any $\tilde{\sigma} \in \Omega$ and any $i \in \Gamma$

$$\left(\sum_{j\in\Gamma} J_{ij}\tilde{\sigma}_j\right)^2 < C < \infty, \qquad E\text{-a.e.} \tag{4}$$

Hence, the finite-volume measures

$$\mu^{\tilde{\sigma}}_{\Lambda,J} \equiv \mu^{\tilde{\sigma}}_{\Lambda}(\cdot) := \delta_{\tilde{\sigma}} \frac{\mu_{0\Lambda}(e^{-\beta H}\,\cdot)}{\mu_{0\Lambda}(e^{-\beta H})}, \qquad \Lambda \in \mathscr{F} \tag{5}$$

(where $\delta_{\tilde{\sigma}}$ is the point measure, $\mu_{0\Lambda}$ is the restriction of μ_0 to the spins in Λ, $\mathscr{F}$ all bounded sets in Γ) are well defined E-a.e. on any countable union $\tilde{\Omega}$ of sets of the form

$$\Omega_{\tilde{\sigma}} := \bigcup_{\Lambda\in\mathscr{F}} \{\sigma\colon \sigma_i = \tilde{\sigma}_i,\, i \in \Lambda^c\} \tag{6}$$

Moreover, for any J from a set of E-measure 1 a family $\tilde{\mathscr{E}}_J \equiv \{\mu^{\sigma}_{\Lambda,J}\colon \sigma \in \tilde{\Omega},\, \Lambda \in \mathscr{F}\}$ forms a local specification. Unfortunately, this specification is too poor for us. For example, if $N = 1$ (an Ising spin glass), then $\tilde{\Omega}$ is a countable set. But the existence of the thermodynamic limit for the pressure suggests that the infinite-volume measure for H

$$\mu_J \equiv \lim_{\mathscr{F}_0} \mu_{\Lambda,J} \equiv \lim_{\mathscr{F}_0} \mu^{\sigma\equiv 0}_{\Lambda,J} \tag{7}$$

exists and should be continuous, so $\tilde{\Omega}$ would be of measure zero for μ_J.

We can of course define an extended specification $\mathscr{E}_J$ by enlarging the domain of $\tilde{\mathscr{E}}_J$ to the set

$$\Omega_J := \left\{\sigma \in \Omega\colon \forall i \in \Gamma, \left(\sum_j J_{ij}\sigma_j\right)^2 < \infty\right\} \tag{8}$$

If $\alpha > 1$ in (Cii), then $\Omega_J = \Omega$, but for $\frac{1}{2} < \alpha \leqslant 1$ we shall verify whether Ω_J contains a set of measure 1 for some continuous probability measure μ.

It is shown in Appendix B that for all J from a set of E-measure 1 and for any probability measure μ such that

$$\left(\sum_j J_{ij}\mu\sigma_j\right)^2 < \infty \tag{9a}$$

$$\forall i,j \in \Gamma\ (i \neq j) \quad |\mu(\sigma_i, \sigma_j)| \leqslant C\,|i-j|^{-\tilde{\alpha}d}, \qquad \tilde{\alpha} > 2(1-\alpha) \tag{9b}$$

we have

$$\mu\Omega_J = 1 \tag{10}$$

[Obviously the set of continuous probability measures satisfying (9a), (9b) is nonempty.] Note that (9) implies (10) for a fixed set of E measure 1. [Given a probability measure ν on (Ω, Σ) one can always find a set $\mathbb{J}_\nu$ of E-measure 1 such that for any $J \in \mathbb{J}_\nu$, $(\sum_j J_{ij}\sigma_j)^2 < \infty$, ν-a.e.] As we will explain in Proposition 2 (see also Refs. 1 and 2), one may expect that the infinite-volume measure for H has at most the cluster property with the same decay as interaction. From this, using (9b), we get

$$\tilde{\alpha} = \alpha \Rightarrow \alpha > \tfrac{2}{3}$$

We expect that this fact will have some important consequences.

Given a local specification $\mathscr{E}_J = \{\mu^\sigma_{\Lambda,J}; \sigma \in \Omega_J, \Lambda \in \mathscr{F}\}$ we define a Gibbs measure μ_J for this specification by the DLR equation

$$\mu_J(\mu^\sigma_{\Lambda,J}(F)) = \mu_J(F)$$

[for any bounded measurable function F on (Ω, Σ)]. The set of all Gibbs measures for $\mathscr{E}_J$ is denoted by $\mathscr{G}(\mathscr{E}_J)$ and the set of its extremal points (pure Gibbs states) by $\partial\mathscr{G}(\mathscr{E}_J)$.

Suppose that for any J from a set $\mathbb{J}$, $E(\mathbb{J}) = 1$, the set $\mathscr{G}(\mathscr{E}_J)$ is nonempty. [Then also the set of its extremal points $\partial\mathscr{G}(\mathscr{E}_J)$ is nonempty; see Ref. 12.] We say that the measures $\mu_J \in \mathscr{G}(\mathscr{E}_J)$ are *weakly unique* if for any probability measure ν on (Ω, Σ)

$$\lim_{\mathscr{F}_0} \mu^{\tilde{\sigma}}_{\Lambda,J} = \mu_J, \qquad \nu\text{-a.e.} \tag{11}$$

on a set $\mathbb{J}_\nu$ of E-measure one.

It was argued in Ref. 10 that this notion of uniqueness is sufficient for physics, since the boundary conditions $\tilde{\sigma}$ should be considered as part of the experimental setup and so are independent of J. We say that the measures $\mu_J \in \mathscr{G}(\mathscr{E}_J)$ are *(strongly) unique* if

$$\#\mathscr{G}(\mathscr{E}_J) = 1 \tag{12}$$

for any $J \in \mathbb{J}$, $E(\mathbb{J}) = 1$.

One may ask what the relation is between these two notions of uniqueness on the set $\mathbb{J}_\nu \cap \mathbb{J}$.

By definition of specifications $\mathscr{E}_J$ we have that there is a set $\Omega_\nu \subset \Omega$ of ν-measure 1 such that

$$\forall J \in \mathbb{J}_\nu \cap \mathbb{J}, \qquad \Omega_\nu \subset \Omega_J \tag{13}$$

In general, however, we can have

$$\mu_J(\Omega_\nu)=0 \tag{14}$$

so weak uniqueness does not necessarily imply extremality of μ_J. On the other hand, the strongly unique Gibbs measure is of course weakly unique and necessarily extremal. (We expect that for $\frac{2}{3}<\alpha$ the weak uniqueness at high temperatures implies the strong one.) For $\alpha>1$ the existence and weak uniqueness problems for spin glasses (as well as the clustering properties of Gibbs measures) have been studied in Refs. 5–11 (see also the references cited therein). At high temperatures one also has strong uniqueness for this case of α values.

The general case $\alpha>\frac{1}{2}$ in the high-temperature region has been solved in Refs. 1 and 2. The following proposition is essentially contained there. (Additionally, we verify here that the infinite-volume measures satisfy the corresponding DLR equations and are extremal Gibbs measures.)

Proposition 1. If E satisfies condition (C) with $\alpha>\frac{1}{2}$, then for all $0<\beta<\beta_0$ with $\beta_0>0$ sufficiently small, the limits

$$\mu_J=\lim_{\mathscr{F}_0}\mu_{\Lambda,J} \tag{15}$$

exist and are weakly unique, extremal Gibbs measures for $\mathscr{E}_J$ (respectively), E-a.e.

To give an idea of the proof, let as consider an Ising spin glass and assume that μ_0 and E are symmetric (for the general case see Ref. 2). We start from *the weak uniqueness* problem. Let ν be a probability measure on (Ω, Σ).

It is sufficient to show that

$$\lim_{\mathscr{F}_0} E\nu(\mu_\Lambda^{\tilde{\sigma}}\sigma_A-\mu_\Lambda\sigma_A)^2=0 \tag{16}$$

for any $A\in\mathscr{F}$, where $\sigma_A\equiv\prod_{i\in A}\sigma_i$ and we have omitted the integration variables J in the notation of finite-volume measures. For notational simplicity we will take $\nu=\delta_{\tilde{\sigma}}$, since in the general case the proof goes analogously.

A spin σ_i, $i\in\Lambda$, feels the external conditions $\tilde{\sigma}\in\Omega$:

1. Directly, through the interaction by the couplings $\{J_{ij}, j\in\Lambda^c\}$.
2. Indirectly, through the interaction with other spins σ_l, $l\in\Lambda$, which interact with the external configuration $\tilde{\sigma}\in\Omega$.

To begin, we separate and estimate the influence on expectation of the function

$$I_A \equiv (\mu_A^{\tilde{\sigma}} \sigma_A - \mu_A \sigma_A)^2 \tag{17}$$

due to the direct interaction of σ_i, $i \in \Lambda$, with spins outside Λ.

By Taylor expansion with remainder of I_A with respect to some coupling J_{ij}, $j \in \Lambda^c$, we get

$$\begin{aligned} EI_A = EI_{A/s_{ij}=0} + \beta E J_{ij}\, \partial_{ij} I_{A/s_{ij}=0} \\ + \beta^2 E \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij}\, J_{ij}^2\, \partial_{ij}^2 I_{A/s'_{ij}} \end{aligned} \tag{18}$$

where we defined interpolating $I_{A/s_{ij}}$ as in (17) but with $s_{ij}J_{ij}$, $s_{ij} \in [0, 1]$, instead of J_{ij}, and $\partial_{ij} \equiv d/d\beta J_{ij} s_{ij}$. From the mean-zero condition (Ci) the second term on the rhs of (18) vanishes.

Next we expand $EI_{A/s_{ij}=0}$ with respect to another $J_{ij'}$, $j' \in \Lambda^c$. Successive application of this procedure gives us

$$EI_A = EI_A(K_{i\Lambda^c}) + \beta^2 \sum_{j \in \Lambda^c} E \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij}\, J_{ij}^2\, \partial_{ij}^2 I_A(\mathscr{C}_{ij}) \tag{19}$$

where we denoted the following conditions:

$$\begin{aligned} K_{i\Lambda^c} &\equiv \{ s_{ij} = 0 \quad \forall j \in \Lambda^c \} \\ \mathscr{C}_{ij} &\equiv \{ s_{il}, \quad l < j; \quad s'_{ij} \} \end{aligned} \tag{20}$$

for a lexicographic order $<$ in Γ.

Since (by our assumption about the values of single spin)

$$|\partial_{ij}^2 I_A(\mathscr{C}_{ij})| \leqslant 20 \tag{21}$$

so using condition (Cii), we get

$$EI_A \leqslant EI_A(K_{i\Lambda^c}) + 20\beta^2\gamma^2 \sum_{j \in \Lambda^c} |i-j|^{-2\alpha d} \tag{22}$$

If $\alpha > \frac{1}{2}$, the series on the rhs of (22) converges and is of order $d(i, \partial\Lambda)^{-(2\alpha-1)d}$.

Now we need to estimate the first term from the rhs of (22), in which the spin σ_i depends on external conditions only indirectly. Expanding successively the integrand $I_A(K_{i\Lambda^c})$ into Taylor series with remainder with respect to J_{ik}, $k \in \Lambda \backslash \{i\}$, we get

$$\begin{aligned} EI_A(K_{i\Lambda^c}) = EI_A(K_{i\Gamma}) + \beta^2 \sum_{k \in \Lambda \backslash \{i\}} |i-k|^{-2\alpha d}\, E \int_0^1 ds_{ik} \int_0^{ik} ds'_{ik}\, J_{ik}^2 \\ \times \partial_{ik}^2 I_A(\mathscr{C}_{ik}) \end{aligned} \tag{23}$$

where $\mathscr{C}_{ik} \equiv \{ s_{ik} = 0, l \in \Lambda^c \text{ or } l < k;\ s'_{ik} \}$

The first term on the rhs of (23) vanishes if μ_0 is symmetric (otherwise it depends only on expectations of $\sigma_{A\setminus i}$ and can be treated analogously starting now from some $i' \in A\setminus i$). The key role in our estimations is played by the following inequality:

$$|\partial^2_{ik} I_A| \leqslant C(I_A + I_{A\cup\{i,k\}} + I_{\{i,k\}}) \tag{24}$$

with a constant $C > 0$ independent of any parameter of our model. This is a purely algebraic inequality.

Using (22)–(24), we get the bound

$$EI_A \leqslant 20\beta^2\gamma^2 \sum_{j\in A^c} |i-j|^{-2\alpha d} + C\beta^2 \sum_{k\in A} E\int_0^1 ds_{ik} \int_0^{s_{ik}} ds'_{ik} J^2_{ik} [I_A + I_{A\cup\{ik\}} + I_{\{ik\}}]_{|\mathscr{C}_{ik}} \tag{25}$$

Since each term of the form $I_{B/\mathscr{C}_{ik}}$ in the integrals in the second series on the rhs of (25) has the same structure as the starting one I_A, we can apply to them the same procedure (19)–(25) starting from the point $k \in A$, respectively.

By iteration of the above steps we generate our upper expansion. The control over our upper expansion gives us the following facts:

1. In each step we get a small factor β^2.
2. The number of terms grows only exponentially, since each factor I_B produces a constant number of terms of the same structure.
3. Due to condition (C), we get summable factors $|i-k|^{-2\alpha d}$.
4. If we expand in the direction of internal line $J_{kk'}$, k, $k' \in A$, which was interpolated in the preceding step, we get multiple integrals with respect to $s_{kk'}$. The $2n$ of such multiple integrals produces the factor $[(2n)!]^{-1}$, which cancels the corresponding $(2n)!$ factor coming from $EJ^{2n}_{kk'}$ [by our ($\mathscr{C}$ii)].

See Refs. 1 and 2 for details.

In general, if E is nonsymmetric, we always expand the first time with respect to a $J_{ik'}$ as above, but the next time (in the same direction) we expand only to the first order and use, analogously to (24), the inequality

$$|\partial_{ik} I_A| \leqslant C'(I_A + I_{A\cup\{ik\}} + I_{\{ik\}}) \tag{26}$$

with a numerical constant $C' > 0$.

By resummation of our upper expansion, if $0<\beta<\beta_0$, for $\beta_0>0$ sufficiently small, we get the bound

$$\begin{aligned} E(\mu_A^{\tilde{\sigma}}\sigma_A-\mu_A\sigma_A)^2 &\leqslant C\sum_{i\in A}\sum_{j\in A^c}|i-j|^{-2\alpha d} \\ &\leqslant C\,|A|\;d(A,\partial A)^{-(2\alpha-1)d} \end{aligned} \tag{27}$$

with a constant $C>0$ independent of A. This ends the proof of weak uniqueness.

To prove the *existence of infinite-volume measures,* we consider the quantities $E(\mu_{A_2}\sigma_A-\mu_{A_1}\sigma_A)^2$ with $A\in\mathscr{F}$, $A_1\subset A_2$, A_1, $A_2\in\mathscr{F}_0$.

Proceeding analogously as in the proof of weak uniqueness, first we estimate the influence of spins in $A_2\backslash A_1$ on the spins in A due to the direct interaction through the couplings J_{ij}, $i\in A$, $j\in A_2\backslash A_1$. Then we consider the indirect dependence due to interaction with other spins in A_1 that are connected with external (to A_1) spins.

The resummation of the upper expansion gives the following bound:

$$E(\mu_{A_2}\sigma_A-\mu_{A_1}\sigma_A)^2\leqslant C\,|A|\;d(A,\partial A_1)^{-(2\alpha-1)d} \tag{28}$$

with $C>0$ independent of A_1, A_2, if $0<\beta<\beta_0$ for sufficiently small $\beta_0>0$.

The estimation (28) implies the existence of limits

$$\mu_J=\lim_{\mathscr{F}_0}\mu_{A,J} \tag{29}$$

for some subsequence $\mathscr{F}_0$ of bounded sets and E-a.a. J. Now we want to verify that μ_J *are Gibbs measures for* $\mathscr{E}_J$. First we note that for any $i\in\Gamma$,

$$E\mu_J\Bigg(\sum_{\substack{j\in\Gamma\\ j\neq i}}J_{ij}\sigma_j\Bigg)^2<\infty \tag{30}$$

which easily follows with the use of (Cii) from

$$\begin{aligned} &|EJ_{ij_1}J_{ij_2}\mu_J\sigma_{j_1}\sigma_{j_2}| \\ &\quad=\left|\beta^2\int_0^1 ds_{ij_1}\int_0^1 ds_{ij_2}\;EJ^2_{ij_1}J^2_{ij_2}\,\partial_{ij_1}\,\partial_{ij_2}(\mu_J\sigma_{j_1}\sigma_{j_2})^2\right| \\ &\quad\leqslant 20\beta^2EJ^2_{ij_1}J^2_{ij_2};\qquad j_1\neq j_2 \end{aligned} \tag{31}$$

[Here Taylor expansion with remainder and (Ci) have been used.] From (30) we see that the measures $\{\mu^{\tilde{\sigma}}_{AJ}\}_{A\in\mathscr{F}}$ are well defined μ_J-a.e. (for E-a.a. J).

We now show

$$\mu_J \mu_{\Lambda,J}^{\tilde{\sigma}} \sigma_A = \mu_J \sigma_A, \qquad \forall \Lambda \in \mathscr{F} \tag{32}$$

i.e., $\mu_J \in \mathscr{G}(\mathscr{E}_J)$.

From (30) for any $\varepsilon > 0$ there is $\Lambda_1 \in \mathscr{F}_0$, $\Lambda \subset \Lambda_1$, such that

$$E\mu(\mu_\Lambda^{\tilde{\sigma}} \sigma_A - \mu_\Lambda^{\tilde{\sigma}_{\Lambda_1}} \sigma_A)^2 < \varepsilon \tag{33}$$

where $\tilde{\sigma}_{\Lambda_1} \equiv \{(\tilde{\sigma}_{\Lambda_1})_i = \tilde{\sigma}_i \text{ if } i \in \Lambda_1 \text{ and } 0 \text{ otherwise}\}$, and we have omitted the indices J.

By definition of μ_J, it is enough to show that for sufficiently large $\Lambda_2 \in \mathscr{F}_0$, $\Lambda_1 \subset \Lambda_2$, one has

$$E(\mu_{\Lambda_2} \mu_\Lambda^{\tilde{\sigma}_{\Lambda_1}} \sigma_A - \mu_{\Lambda_2} \sigma_A)^2 \leqslant \varepsilon \tag{34}$$

Since [with $\tilde{\sigma}(s) \equiv s\tilde{\sigma}_{\Lambda_2} + (1-s)\,\tilde{\sigma}_{\Lambda_1}$]

$$\begin{aligned} &E(\mu_{\Lambda_2} \mu_\Lambda^{\tilde{\sigma}_{\Lambda_1}} \sigma_A - \mu_{\Lambda_2} \sigma_A)^2 \\ &\quad\leqslant E\mu_{\Lambda_2}(\mu_\Lambda^{\tilde{\sigma}_{\Lambda_1}} \sigma_A - \mu_\Lambda^{\tilde{\sigma}_{\Lambda_2}} \sigma_A)^2 \\ &\quad= E\mu_{\Lambda_2}\left(\int_0^1 ds \frac{d}{ds} (\mu_\Lambda^{\tilde{\sigma}(s)} \sigma_A)^2\right) \\ &\quad= E\mu_{\Lambda_2}\left(\int_0^1 ds\, \mu_\Lambda^{\tilde{\sigma}(s)} \left(\sigma_A, \sum_{i\in\Lambda} \sum_{j \in \Lambda_2 \setminus \Lambda_1} \sigma_i J_{ij} \tilde{\sigma}_j\right)\right)^2 \\ &\quad\leqslant C 2^{|\Lambda|} \sum_{i \in \Lambda} E\mu_{\Lambda_2}\left(\sum_{j\in\Lambda_2\setminus\Lambda_1} J_{ij}\tilde{\sigma}_j\right)^2 \\ &\quad\leqslant C' 2^{|\Lambda|}\, |\Lambda|\, d(\Lambda, \partial\Lambda_1)^{-(2\alpha-1)d} \end{aligned} \tag{35}$$

by the same arguments as in the proof of (30), so we get (34). This ends the proof of (32). Note that we used here the fact that $\{\mu_{\Lambda J}\}_{\Lambda \in \mathscr{F}_0}$ converges and the limit μ_J is twice (jointly) differentiable with respect to the couplings J_{ij} $(i, j \in \Gamma)$, but *the assumption* $0 < \beta < \beta_0$ *with* $\beta_0 > 0$ *sufficiently small is not used.*

Extremality. In order to show that $\mu_J \in \partial\mathscr{G}(\mathscr{E}_J)$, E-a.e., it is sufficient to prove that

$$\lim_{\mathscr{F}_0} E\mu(\mu_\Lambda^{\tilde{\sigma}} \sigma_A - \mu_\Lambda \sigma_A)^2 = 0 \tag{36}$$

for any $A \in \mathscr{F}$. For this we use a slightly more complicated, but analogous as in the proof of weak uniqueness, upper expansion. The complications come from the fact that now μ is J-dependent.

Let $i \in \Lambda$. By successive Taylor expansion with remainder with respect to J_{ij}, $j \in \Lambda^c$, we get

$$\begin{aligned}
&E\mu(\mu_\Lambda^{\tilde{\sigma}}\sigma_\Lambda - \mu_\Lambda\sigma_\Lambda)^2 \\
&\quad \equiv E\mu I_\Lambda = E\mu[I_\Lambda(K_{i\Lambda^c})] \\
&\qquad + \beta \sum_{j \in \Lambda^c} E[J_{ij}\mu(\partial_{ij} I_\Lambda(K_{ij}))] \\
&\qquad + \beta^2 \sum_{j \in \Lambda^c} E\left[J_{ij}^2 \mu \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij}\, \partial_{ij}^2 I_\Lambda(\mathscr{C}_{ij})\right] \qquad (37)
\end{aligned}$$

where $K_{ij} \equiv \{s_{il} = 0,\ l \lesssim j\}$, $\mathscr{C}_{ij} \equiv \{s_{il}, l < j; s'_{ij}\}$. Now the second sum does not vanish, since μ depends on J. We have from (Ci)

$$\begin{aligned}
&|\beta E J_{ij} \mu[\partial_{ij} I_\Lambda(K_{ij})]| \\
&\quad = \left| \beta^2 E J_{ij}^2 \int_0^1 ds_{ij} (\partial_{ij}\mu)_{/s_{ij}} [\partial_{ij} I_\Lambda(K_{ij})] \right| \\
&\quad \leqslant 32\beta^2 E J_{ij}^2 \qquad (38)
\end{aligned}$$

Hence both series on the rhs of (37) are bounded by

$$C_1 \sum_{j \in \Lambda^c} |i-j|^{-2\alpha d} \sim o(d(\Lambda, \partial\Lambda)^{-(2\alpha-1)d}), \qquad \text{with} \quad C_1 > 0$$

independent of Λ.

To estimate the first term on the rhs of (37), we make the expansion step with respect to the couplings in Λ. We have

$$\begin{aligned}
E\mu I_\Lambda(K_{i\Lambda^c}) = E\mu I_\Lambda(K_{i\Gamma}) &+ \beta \sum_{\substack{k \in \Lambda \\ k \neq i}} E J_{ik} \mu(\partial_{ik} I_\Lambda(K_{ik})) \\
&+ \beta^2 \sum_{\substack{k \in \Lambda \\ k \neq i}} E J_{ik}^2 \int_0^1 ds_{ik} \int_0^{s_{ik}} ds'_{ik}\, \mu(\partial_{ik}^2 I_\Lambda(\mathscr{C}_{ik})) \qquad (39)
\end{aligned}$$

with

$$K_{ik} \equiv K_{i\Lambda^c} \cup \{s_{il} = 0, l \lesssim k\}$$
$$\mathscr{C}_{ik} \equiv K_{i\Lambda^c} \cup \{s_{il} = 0, l < k, s'_{ik}\}$$

The first term on the rhs of (39) vanishes if μ_0 is symmetric. (Otherwise it depends on expectations of $\sigma_{\Lambda \setminus i}$ and should be expanded further. Then we have some induction in volume of Λ.) To estimate the first series from the rhs of (39) we note that

$$\begin{aligned}
&|\beta E J_{ik}\mu(\partial_{ik} I_\Lambda(K_{ik}))| \\
&\quad = \left|\beta^2 E J_{ik}^2 \int_0^1 ds_{ik}(\partial_{ik}\mu_{/s_{ik}})(\partial_{ik} I_\Lambda(K_{ik}))\right| \\
&\quad \leqslant 2\beta^2 E J_{ik}^2 \int_0^1 ds_{ik}\, \mu_{/s_{ik}}\, |\partial_{ik} I_\Lambda(K_{ik})| \qquad (40)
\end{aligned}$$

Then, using the inequalities (24) and (26), we get the estimate

$$\begin{aligned}
E\mu I_\Lambda(K_{i\Lambda^c}) \leqslant \beta^2 \cdot C \sum_{\substack{k\in\Lambda\\ k\neq i}} E J_{ik}^2 \Bigg[\int_0^1 ds_{ik}\, \mu_{/s_{ik}}(I_\Lambda(K_{ik}) \\
+ I_{\Lambda\cup\{i,k\}}(K_{ik}) + I_{\{i,k\}}(K_{ik})) \\
+ \int_0^1 ds_{ik} \int_0^{s_{ik}} ds'_{ik}\, \mu \Big(I_\Lambda(\mathscr{C}_{ik}) \\
+ I_{\Lambda\cup\{ik\}}(\mathscr{C}_{ik}) + I_{\{i,k\}}(\mathscr{C}_{ik})\Big)\Bigg] \qquad (41)
\end{aligned}$$

Hence we get our main inequality,

$$E\mu I_\Lambda \leqslant C_1 \sum_{j\in\Lambda} |i-j|^{-2\alpha d} + \text{rhs of (41)} \qquad (42)$$

Since the second term contains factors of the form I_B of the same structure as the starting one I_Λ, we can apply to them the same procedure (starting from the point k, respectively). In this manner, we generate an adequate upper expansion. The principle of control of this expansion is the same as in the case of the proof of weak uniqueness. However, now we require β_0 to be smaller because in each step we generate twice as many terms. After resummation of our upper expansion we get the bound

$$E\mu(\mu_\Lambda^{\bar\sigma}\sigma_A - \mu_\Lambda\sigma_A)^2 \leqslant C\,|A|\; d(A, \partial\Lambda)^{-(2\alpha-1)d} \qquad (43)$$

which ends the proof of extremality.

Remarks. Note that in the case of the Ising spin glass, since $\sigma_i^2 = 1$, we can make several simplifications in the organization of the full upper expansion; see Refs. 1 and 2.

Let us stress that in our proofs we have not used the translation invariance of E, but only the condition (C).

As a simple consequence of Proposition 1, we have the following result:

Corollary 1. For any $\Lambda \in \mathscr{F}$ and any bounded measurable function F on (Ω, Σ) we have

$$E(\mu(\sigma_A, F))^2 \leqslant C \, \|F\|_\infty^2 \, |A| \, d(A, F)^{-(2\alpha - 1)d} \tag{44}$$

Proof. Let $\Lambda \in \mathscr{F}$, $A \subset \Lambda$, be the largest set such that $F \in \Sigma_{\Lambda^c}$. Then we have

$$\begin{aligned} E(\mu(\sigma_A, F))^2 &= E(\mu((\mu_\Lambda^{\tilde{\sigma}} \sigma_A - \mu_\Lambda \sigma_A), F))^2 \\ &\leqslant 4 \, \|F\|_\infty^2 \, E\mu(\mu_\Lambda^{\tilde{\sigma}} \sigma_A - \mu_\Lambda \sigma_A)^2 \end{aligned} \tag{45}$$

Hence, using Proposition 1, we get (44).

In Refs. 1 and 2 we have proven (for Ising and N-vector spin glasses) a more precise result for clustering, namely:

Proposition 2 (Clustering). Let E satisfy (C) with $\alpha > \frac{1}{2}$ and $0 < \beta < \beta_0$ with $\beta_0 > 0$ sufficiently small. Then for any bounded set $A \subset \Gamma$ and any bounded function $F \in \Sigma_B$, $B \equiv \operatorname{supp} F \subset \Gamma \backslash A$,

$$E(\mu(\sigma_A, F))^2 \leqslant C \, \|F\|_\infty^2 \sum_{\substack{i \in A \\ j \in B}} |i - j|^{-2\alpha d} \tag{46}$$

with a constant $C > 0$ independent of B and any monomial σ_A in spin components $\sigma_i^{m(i)}$, $i \in A$.

Sketch of the Proof. In order to give an idea of the proof, let us observe that there is a hierarchy of correlations between spins in the sets A and B. First, there is the direct interaction between spins in both regions. If we remove this direct interaction, the spins in A and B are still correlated, since they interact directly with other spins σ_{k_1}, $k_1 \notin A \cup B$. After removing the direct interaction of a spin σ_{k_1}, $k_1 \notin A \cup B$, with those in B, the spins in A and B still can be correlated. This holds if there are other spins $\sigma_{k_2} \notin A \cup B \cup \{k_1\}$ that interact directly with σ_{k_1} and spins from B. And so on....

For simplicity of notation let us consider only an Ising spin glass with μ_0 and E symmetric; for the general case see Ref. 2.

Let us take $i \in A$. We first estimate the correlations of σ_i with spins in B due to direct interactions. By successive Taylor expansion with remainder we have

$$EI_{A,B} = EI_{A,B}(K_{iB}) + \beta^2 \sum_{j \in B} EJ_{ij}^2 \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij} \, \partial_{ij}^2 I_{A,B}(\mathscr{C}_{ij}) \tag{47}$$

where

$$I_{A,B} \equiv \mu(\sigma_A, F)^2$$
$$K_{iB} \equiv \{s_{ij} = 0, \forall j \in B\}$$
$$\mathscr{C}_{ij} \equiv \{s_{il} = 0, l < j; s'_{ij}\}$$

The second term on the rhs of (47) is estimated by

$$a \, \|F\|_\infty^2 \sum_{j \in B} |i-j|^{-2\alpha d}$$

[from (Cii)].

To analyze the second-order correlations (for σ_i), we expand successively the first term on the rhs of (47):

$$EI_{A,B}(K_{iB}) = EI_{A\setminus\{i\},B}(K_{i\Gamma})(\mu_0\sigma_i)^2 + \beta^2 \sum_{k \in \Gamma\setminus B \cup \{i\}} EJ_{ik}^2 \int_0^1 ds_{ik} \int_0^{s_{ik}} ds'_{ik} \, \partial_{ik}^2 I_{A,B}(\mathscr{C}_{ik}) \tag{48}$$

The first term on the rhs of the above equality vanishes if μ_0 is symmetric (otherwise it should be expanded further analogously). To continue our upper expansion, we use the following (algebraic) inequality:

$$|\partial_{ik}^2 I_{A,B}| \leqslant C(I_{A,B} + I_{A \cup \{i,k\},B} + I_{\{i,k\},B}) \tag{49}$$

with a numerical constant $C > 0$.

From (47)–(49) we get the inequality

$$EI_{A,B} \leqslant a \, \|F\|_\infty^2 \sum_{j \in B} |i-j|^{-2\alpha d} + C\beta^2 \sum_{k \in \Gamma\setminus B \cup \{i\}} EJ_{ik}^2 \int_0^1 ds_{ik} \int_0^{s_{ik}} ds'_{ik} \, (I_{A,B} + I_{A \cup \{i,k\},B} + I_{\{i,k\},B})(\mathscr{C}_{ik}) \tag{50}$$

which can be used for the generation of the upper expansion in the considered case. The control over this expansion is based on analogous facts as previously. The resummation of our upper expansion gives (46). Note also that if $\beta_0 > 0$, is sufficiently small, we can use (49) to generate a series that gives the estimation of the decay of correlations from below (see Ref. 2).

The above-described upper expansion method also can be applied to the investigation of nonrandom systems. Then in general we make the

Taylor expansion of $(\mu_A^{\tilde{\sigma}}\sigma_A - \mu_A\sigma_A)^2$ or $\mu(\sigma_A, \sigma_B)^2$ with respect to the couplings J_{ij} only to the first order. In particular this gives for $J_{ij} \sim |i-j|^{-\alpha d}$, $\alpha > 1$, the decay of correlations $\mu(\sigma_A, \sigma_B) \sim d(A, B)^{-\alpha d/2}$ (for any bounded sets A, $B \subset \Gamma$).

However, in some special cases, when additional information is available (e.g., due to correlation inequalities), we can apply our upper expansion directly to $\mu(\sigma_A, \sigma_B)$. Then we get the decay of correlations exactly as the decay of interaction. Actually we know how to get this result by our method in the general case of nonrandom spin system (with a bounded single spin space).

As an example we give in Appendix C a simple proof of this fact for ferromagnets at high temperatures.

APPENDIX A

It is sufficient to consider as a free measure μ_0 a symmetric product probability measure on (Ω, Σ). Let us take E to be symmetric (for the general case one can use an analogous method).

Proposition A.1. If

$$EJ_{ij} = 0, \qquad \forall i, j \in \Gamma \tag{A.1a}$$

$$|EJ_{ij}^m| \leqslant \gamma^m m! \, |i-j|^{-m\alpha d} \qquad \text{for} \quad m \leqslant 4, \quad \alpha > \tfrac{1}{2}, \gamma > 0 \tag{A.1b}$$

then

$$\lim_{\mathscr{F}} p_{A_n} = p \equiv \lim Ep_{A_n}; \qquad E - \text{a.e.} \tag{A.2}$$

We need the following lemma:

Lemma A.1. If (A.1a) and (A.1b) hold with $m \leqslant 2n$ for $n \in \mathbb{N}$, then

$$E(p_A(J))^n < C_n$$

with $C_n > 0$ independent of A.

Proof. From our assumptions about μ_0 we have for any J

$$p_A(J) \geqslant 0 \tag{A.3}$$

Let us first consider the case $n = 1$. By successive Taylor expansion with remainder, using the mean-zero condition, we get

$$Ep_A(J) = \sum_{(i,j) \subset A} \beta^2 E\left[J_{ij}^2 \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij} \, \partial_{ij}^2 \, p_A(J)_{/\mathscr{C}_{ij}} \right] \tag{A.4}$$

where

$$\mathscr{C}_{ij} \equiv \{s_{kl} = 0, \, (kl) < (i, j); \, s'_{ij}\}$$

(with a lexicographic order $<$ in the space of bonds).

Since

$$|\partial_{ij}^2 p_\Lambda(J)| = |\Lambda|^{-1} |\mu_\Lambda(\sigma_i \cdot \sigma_j, \sigma_i \cdot \sigma_j)| \leqslant 2\,|\Lambda|^{-1} \tag{A.5}$$

so by our condition (A.1b) with $m \leqslant 2$ we get

$$Ep_\Lambda(J) \leqslant 2\beta^2\gamma^2 \sum_{\substack{j\in\Gamma\\ j\neq i}} |i-j|^{-2\alpha d} < \infty \tag{A.6}$$

For general $n \in \mathbb{N}$ we make the expansion as in (A.4) and then use the bound

$$\begin{aligned} |\partial_{ij}^2 p_\Lambda^n| &= |n(n-1)\, p_\Lambda^{n-2} (\partial_{ij} p_\Lambda)^2 + np_\Lambda^{n-1}\, \partial_{ij}^2 p_\Lambda| \\ &\leqslant n(n-1)\, |\Lambda|^{-2}\, p_\Lambda^{n-2} + 2n\, |\Lambda|^{-1}\, p_\Lambda^{n-1} \end{aligned} \tag{A.7}$$

to get

$$\begin{aligned} Ep_\Lambda^n \leqslant \beta^2 \sum_{(i,j)\subset\Lambda} E\Big\{J_{ij}^2 \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij} [n(n-1)\, |\Lambda|^{-2}\, p_\Lambda^{n-2} \\ + 2n\, |\Lambda|^{-1}\, p_\Lambda^{n-1}]_{/\mathscr{C}_{ij}}\Big\} \end{aligned} \tag{A.8}$$

By n applications of this procedure we get the bound

$$\begin{aligned} Ep_\Lambda^n \leqslant \beta^{2n} 2^n n!\, |\Lambda|^{-n} \sum_{(i_1,j_1)\leqslant \cdots \leqslant (i_n,j_n)\subset\Lambda} \cdots \sum \left[\prod_{k=1}^n (2n_k)!\right]^{-1} \\ \times EJ_{i_1 j_1}^2 \cdots J_{i_n j_n}^2 + o(|\Lambda|^{-1}) \end{aligned} \tag{A.9}$$

where $\{n_1,..., n_n\}$ are the numbers of couplings with the same indices and the factor $[\prod_{k=1}^n (2n_k)!]^{-1}$ comes from multiple integrals. From (A.9) we get

$$Ep_\Lambda^n \leqslant 2^{2n+1} \beta^{2n} \gamma^{2n} n! \left(\sum_{\substack{j\in\Gamma\\ j\neq i}} |i-j|^{-2\alpha d}\right)^n + o(|\Lambda|^{-1}) \tag{A.10}$$

This ends the proof of lemma.

Let

$$\Lambda_n := \{|i^l| \leqslant 2^n;\, l = 1,..., d\} \qquad \text{for} \quad n \in \mathbb{N}$$

Let

$$g_n := p(\Lambda_n) - 2^{-d} \sum_{l=1,...,2^d} p(\Lambda_{n-1}^l) \tag{A.11}$$

where $p(\Lambda) \equiv p_\Lambda$, and the union of disjoint cubes Λ_{n-1}^l ($l = 1,..., 2^d$) of side 2^{n-1} is equal to Λ_n.

By Jensen's inequality we see that $g_n \geqslant 0$.

Lemma A.2. If (A.1a) and (A.1b) hold with $m \leqslant 2$, then

$$Eg_n \leqslant C2^{-n(2\alpha-1)d}, \qquad \forall n \in \mathbb{N} \tag{A.12}$$

with $C > 0$ independent of n.

Proof. Applying the successive Taylor expansion with remainder with respect to the couplings J_{ij}, $i \in \Lambda_{n-1}^l$, $j \in \Lambda_{n-1}^k$, with $l \neq k$, we get

$$0 \leqslant Eg_n = \beta^2 \sum_{\substack{i \in \Lambda_{n-1}^1, j \in \Lambda_{n-1}^k \\ l \neq k; l,k = 1,...,2^d}} E\left(J_{ij}^2 \int_0^1 ds_{ij} \int_0^{s_{ij}} ds'_{ij}\, \partial_{ij}^2 g_{n/\mathscr{C}_{ij}}\right) \tag{A.13}$$

Since

$$|\partial_{ij}^2 g_n| \leqslant 2\, |\Lambda_n|^{-1} \tag{A.14}$$

so we have

$$Eg_n \leqslant 2\beta^2\gamma^2\, |\Lambda_n|^{-1} \sum_{(\cdots)} |i-j|^{-2\alpha d} \tag{A.15}$$

where the summation goes over all (i, j) as in (A.13). The sum on the rhs of (A.15) is of order $C \cdot 2^d(\operatorname{diam} \Lambda_{n-1})^{2(1-\alpha)d}$ with $1 \geqslant \alpha > \frac{1}{2}$, and so from (A.15) the lemma follows.

From Lemma A.2 it easily follows that

$$p = \lim_{n \to \infty} Ep_{\Lambda n} \tag{A.16}$$

exists. Since now

$$E\, |p_{\Lambda n} - Ep_{\Lambda n}| \leqslant 2Eg_n + E \left| 2^{-d} \sum_{l=1,...,2^d} p(\Lambda_{n-1}^l) - E2^{-d} \sum_{l=1,...,2^d} p(\Lambda_{n-1}^l) \right| \tag{A.17}$$

and by iteration

$$E\, |p_{\Lambda n} - Ep_{\Lambda n}| \leqslant 2 \sum_{k=0}^{k(n)} Eg_{n-k} + E\, |S_{k(n)} - ES_{k(n)}| \tag{A.18}$$

with

$$S_{k(n)} \equiv 2^{-k(n)d} \sum_{l=1,\dots,2^{k(n)d}} p(\Lambda^l_{n-k(n)}) \tag{A.19}$$

so, taking $k(n) \to \infty$ as $n \to \infty$ and using (A.12), Lemma A.1, and the law of large numbers, we see that

$$\lim_{n \to \infty} E\,|p_{\Lambda n} - Ep_{\Lambda n}| = 0 \tag{A.20}$$

This, together with (A.16), ends the proof of Proposition A.1 for the sequence $\mathscr{F}_0 \equiv \{\Lambda_n\}_{n \in N}$. The generalization to any Fisher sequence can be done as in Ref. 3.

Remark. Note that under the conditions (A.1a), (A.1b) one can prove the existence of (random) infinite-volume pressure (independent of boundary conditions, but it may be dependent on a sequence $\mathscr{F}_0$) without the assumption about the translational invariance of E.

APPENDIX B

Lemma B.1. If E satisfies (C) with $\alpha > \frac{1}{2}$, then for any set $\mathbb{J}$ of E-measure 1 and any probability measure μ on (Ω, Σ) such that for any $i, k \in \Gamma$

$$\forall J \in \mathbb{J} \qquad \left(\sum_{j \in \Gamma} J_{ij} \mu\sigma_j\right)^2 < \infty \tag{B.1}$$

$$|\mu(\sigma_i, \sigma_k)| \leqslant C[1 + |i-k|]^{-\tilde{\alpha}d}, \qquad 2(1-\alpha) < \tilde{\alpha} \tag{B.2}$$

we have

$$\forall J \in \mathbb{J} \qquad \left(\sum_{j \in \Gamma} J_{ij} \sigma_j\right)^2 < \infty, \qquad \mu\text{-a.e.} \tag{B.3}$$

Remark. In particular, if $\mu\sigma_j = \text{const}$, (B.1) holds on a set of E-measure 1.

Proof. Let $\mathbb{J}$ be a set of such couplings J that for any $i \in \Gamma$ and any $\frac{1}{2} > \delta > 0$ there is a constant $C_\delta \equiv C_\delta(i, J) > 0$ with the property

$$\forall j \in \Gamma \qquad |J_{ij}| < C_\delta\, |i-j|^{-(\alpha-\delta)d} \tag{B.4}$$

and

$$\left(\sum_{j \in \Gamma} J_{ij}\right)^2 < \infty \tag{B.5}$$

Then from (C) we have

$$E(\mathbb{J}) = 1 \tag{B.6}$$

Let $\mathscr{K}(i, r) = \{j\colon |i-j| < r\}$; then

$$\mu\Big(\sum_{j\in\mathscr{K}(i,r)} J_{ij}\sigma_j\Big)^2 = \sum_{j,j'\in\mathscr{K}(i,r)} J_{ij}J_{ij'}\mu(\sigma_j, \sigma_{j'}) + \Big(\sum_{j\in\mathscr{K}(i,r)} J_{ij}\mu\sigma_j\Big)^2 \tag{B.7}$$

From (B.4) and (B.2) we have

$$\Big|\sum_{j,j'\in\mathscr{K}(i,r)} J_{ij}J_{ij'}\mu(\sigma_j, \sigma_{j'})\Big| \leqslant C^2 \sum_{j,j'\in\Gamma} |i-j|^{-(\alpha-\delta)d}\,|i-j'|^{-(\alpha-\delta)d}\,(1+|j-j'|)^{-\tilde{\alpha}d} \tag{B.8}$$

Since

$$\begin{aligned}\sum_{j'\neq i,j} &|i-j'|^{-(\alpha-\delta)d}(1+|j-j'|)^{-\tilde{\alpha}d}\\ &= \sum_{|i-j'|>|j-j'|}(\cdots) + \sum_{|i-j'|\leqslant|j-j'|}(\cdots)\\ &\leqslant C_1|i-j|^{-\varepsilon d}\Big[\sum_{j'\neq j}|j-j'|^{-(\alpha+\tilde{\alpha}-\delta-\varepsilon)d} + \sum_{j'\neq i}|i-j'|^{-(\alpha+\tilde{\alpha}-\delta-\varepsilon)d}\Big]\\ &\leqslant C_2|i-j|^{-\varepsilon d}\end{aligned} \tag{B.9}$$

if

$$(\alpha+\tilde{\alpha}-\delta-\varepsilon) > 1 \tag{B.10a}$$

then the rhs of (B.8) can be estimated by

$$C_3\sum_{j\neq i}|i-j|^{-(\alpha+\varepsilon-\delta)d}$$

which is finite if

$$(\alpha+\varepsilon-\delta) > 1 \tag{B.10b}$$

From (B.10a), (B.10b) we get

$$\tilde{\alpha} > 2(1-\alpha) \tag{B.11}$$

(since $\delta > 0$ can be taken arbitrarily small).

APPENDIX C

We have

$$H = -\sum_{i \neq j} J_{ij} \sigma_i \sigma_j \tag{C.1}$$

with $\sigma_i = \pm 1$, $i \in \Gamma \equiv Z^d$, and

$$J_{ij} \equiv |i-j|^{-\alpha d}, \qquad \alpha > 1 \tag{C.2}$$

for all $i, j \in \Gamma$, $i \neq j$. We define

$$\mu_\Lambda(\cdot) := \delta_{\bar{\sigma} \equiv 0} \frac{\mu_{0\Lambda}(e^{-\beta H} \cdot)}{\mu_{0\Lambda}(e^{-\beta H})} \tag{C.3}$$

with a symmetric free measure μ_0.

Proposition C.1. If $0 < \beta < \beta_0$ for a sufficiently small $\beta_0 > 0$, then

$$\mu(\sigma_i \sigma_j) = o(|i-j|^{-\alpha d}) \tag{C.4}$$

Proof. We have

$$\mu_\Lambda \sigma_i \sigma_j = \mu_\Lambda \sigma_i \sigma_{j/K_{ij}} + \beta\, |i-j|^{-\alpha d} \int_0^1 ds_{ij}\, \partial_{ij}(\mu_\Lambda \sigma_i \sigma_j)_{/\mathscr{C}_{ij}} \tag{C.5}$$

The second term is of the form of a bounded, nonnegative function times $|i-j|^{-\alpha d}$. Hence, we need only to consider the first term. (This term is nonnegative, from GKS, so we have automatically the bound from below.)

We have

$$\mu_\Lambda \sigma_i \sigma_{j/K_{ij}} = \sum_{\substack{k \in \Lambda \\ k \neq i,j}} \beta\, |i-k|^{-\alpha d} \int_0^1 ds_{ik}\, \partial_{ik}(\mu \sigma_i \sigma_j)_{/\mathscr{C}_{ik}} \tag{C.6}$$

where $\mathscr{C}_{ij} = \{s_{il} = 0,\ l < k;\ s_{ik}\}$. (We take j as the first element with respect to a partial order $<$.)

Since from the GKS inequalities

$$\begin{aligned} &\leqslant \partial_{ik} \mu_\Lambda \sigma_i \sigma_j = \mu_\Lambda \sigma_k \sigma_j - \mu_\Lambda \sigma_i \sigma_j \mu_\Lambda \sigma_i \sigma_k \\ &\leqslant \mu_\Lambda \sigma_k \sigma_j \end{aligned} \tag{C.7}$$

so

$$\mu_\Lambda \sigma_i \sigma_{j/K_{ij}} \leqslant \beta \sum_{\substack{k \in \Lambda \\ k \neq i,j}} |i-k|^{-\alpha d} \int_0^1 ds_{ik}\, \mu_\Lambda \sigma_k \sigma_{j/\mathscr{C}_{ik}} \tag{C.8}$$

From (C.5) and (C.8) we get

$$\mu_\Lambda \sigma_i \sigma_j \leqslant \beta \, |i-j|^{-\alpha d} + \beta \sum_{\substack{k \in \Lambda \\ k \neq i,j}} |i-k|^{-\alpha d} \int_0^1 ds_{ik} \, \mu_\Lambda \sigma_k \sigma_{j/\mathscr{C}_{ik}} \tag{C.9}$$

Using this inequality, we can generate the adequate upper expansion. The resummation of this expansion with the use of

$$\sum_{k \neq i,j} |i-k|^{-\alpha d} \, |k-j|^{-\alpha d} \leqslant b \, |i-j|^{-\alpha d}, \qquad \text{for} \quad i \neq j \tag{C.10}$$

for a constant $b > 0$, gives us the bound

$$\mu_\Lambda \sigma_i \sigma_j \leqslant \beta (1 - \beta b)^{-1} \, |i-j|^{-\alpha d} \tag{C.11}$$

if $0 < \beta < \beta_0$ for $0 < \beta_0 < b^{-1}$.

This ends the proof.

Analogous method works if an external magnetic fields is included.

ACKNOWLEDGMENTS

The author would like to thank the physics and mathematics departments of ETH-Zürich for hospitality during the academic year 1985–1986, as well as Prof. J. Fröhlich for the pleasure of collaboration.

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